

CALDERÓN-ZYGMUND OPERATORS WITH NON-DIAGONAL SINGULARITY

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ABSTRACT. In this paper, we introduce a class of singular integral operators which generalize Calderón-Zygmund operators to the more general case, where the set of singular points of the kernel need not to be the diagonal, but instead, it can be a general hyper curve. We show that such operators have similar properties as ordinary Calderón-Zygmund operators. In particular, we prove that they are of weak-type $(1, 1)$ and strong type (p, p) for $1 < p < \infty$.

1. INTRODUCTION

We say that T is a Calderón-Zygmund operator if T is a continuous linear operator maps $C_c^\infty(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n)$ that extends to a bounded operator on $L^2(\mathbb{R}^n)$, and whose distribution kernel K coincides with a function $K(x, y)$ defined away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$(1.1) \quad |K(x, y)| \leq \frac{A}{|x - y|^n},$$

$$(1.2) \quad |K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \leq \frac{A|x - z|^\varepsilon}{|x - y|^{n+\varepsilon}},$$

hold for some $A > 0, \varepsilon > 0$ whenever $|x - y| > 2|x - z|$ and for $f \in C_c^\infty(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy, \quad x \notin \text{supp}(f).$$

Using the Calderón-Zygmund decomposition, one can prove that such an operator is of weak type $(1, 1)$ and consequently strong type (p, p) . Therefore, the classical Calderón-Zygmund theory is a powerful tool in many aspects of harmonic analysis and partial differential equations [6, 7, 10, 11, 13, 18, 23–28]. And it has been widely studied in various directions, e.g., see [1–5, 8, 9, 12] and references therein. We refer to [14–17, 19–23] for some recent advances of the Calderón-Zygmund theory.

Note that the singularity of the kernel K lies in the diagonal $x = y$. In this paper, we generalize the Calderón-Zygmund operator to the more general case, where the set of singular points of the kernel K can be general hyper curves.

We call Γ a *standard hyper curve* in $\mathbb{R}^n \times \mathbb{R}^n$ and denote it by $\Gamma \in SHC$ if Γ is the union of r hyper curves Γ_i in $\mathbb{R}^n \times \mathbb{R}^n$, $1 \leq i \leq r$, and satisfies the followings,

- (i) $\Gamma_i = \{(x, \gamma_i(x)) : x \in \mathcal{D}_i\}$, where γ_i is a mapping from a closed domain $\mathcal{D}_i \subset \mathbb{R}^n$ to \mathbb{R}^n .
- (ii) Each γ_i is differentiable and the Jacobian $J_{\gamma_i}(x)$ of γ_i has no zero in the interior of \mathcal{D}_i .
- (iii) $\gamma_i(x) = \gamma_{i'}(x)$ has at most finitely many solutions for $1 \leq i < i' \leq r$.

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(iv) Both γ_i and γ_i^{-1} satisfy the Lipschitz condition, i.e., there exists some positive constant $c_\gamma > 1$ such that for any $x, x' \in \mathcal{D}_i$ and $y, y' \in \gamma_i(\mathcal{D}_i)$, $1 \leq i \leq r$,

$$(1.3) \quad |\gamma_i(x) - \gamma_i(x')| \leq c_\gamma |x - x'|, \quad |\gamma_i^{-1}(y) - \gamma_i^{-1}(y')| \leq c_\gamma |y - y'|.$$

Note that \mathcal{D}_i might be the same for different i . The set SHC consists of many types of hyper curves, e.g., see Examples 2.5 and 2.6. For simplicity, we also use γ to denote the multi-valued function $\gamma(x) := \{\gamma_i(x) : \mathcal{D}_i \ni x, 1 \leq i \leq r\}$.

For any set $E \subset \mathbb{R}^n$, let $\gamma^{-1}(E) = \bigcup_{i=1}^r \gamma_i^{-1}(E) = \bigcup_{i=1}^r \{x : \gamma_i(x) \in E\}$.

Let $\rho(x, y) = \min \rho_i(x, y)$, where $\rho_i(x, y)$ is the distance from (x, y) to Γ_i , that is,

$$\rho_i(x, y) = \inf_{(x', y') \in \Gamma_i} |(x, y) - (x', y')|.$$

Now we introduce a class of generalized Calderón-Zygmund operators CZO_γ , for which the singularity of the kernel K lies in a standard hyper curve Γ .

Definition 1.1. Let $\Gamma \in SHC$ be a standard hyper curve in $\mathbb{R}^n \times \mathbb{R}^n$. We call $T \in CZO_\gamma$ if T is a continuous linear operator maps $C_c^\infty(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n)$ that extends to a bounded operator on $L^2(\mathbb{R}^n)$, and whose distribution kernel K coincides with a function $K(x, y)$ defined on Γ^c such that

$$(1.4) \quad |K(x, y)| \leq \frac{A}{\rho(x, y)^n},$$

$$(1.5) \quad |K(x, y) - K(x, y')| \leq \frac{A|y - y'|^\delta}{\rho(x, y)^{n+\delta}}, \quad |y - y'| \leq \frac{1}{2}\rho(x, y),$$

$$(1.6) \quad |K(x, y) - K(x', y)| \leq \frac{A|x - x'|^\delta}{\rho(x, y)^{n+\delta}}, \quad |x - x'| \leq \frac{1}{2}\rho(x, y),$$

hold for some $A, \delta > 0$ and for $f \in C_c^\infty(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy, \quad x \notin \gamma^{-1}(\text{supp } f).$$

It is easy to see that if Γ is defined by $y = \gamma(x)$, where γ is an invertible transform from \mathbb{R}^n to \mathbb{R}^n , then we can make T an ordinary Calderón-Zygmund operator with a simple change of variables. However, if γ is not invertible, e.g., Γ is a closed hyper curve, then the change of variables does not work (see Examples 2.5 and 2.6). In other words, CZO_γ is a proper generalization of ordinary Calderón-Zygmund Operators.

We show that operators in CZO_γ have similar properties as ordinary Calderón-Zygmund operators. In Section 2, we consider the truncated kernels and operators and give a formula for the difference between two operators which share the same kernel. In Section 3, we show that every operator in CZO_γ is of weak-type $(1, 1)$ and strong type (p, p) for $1 < p < \infty$.

Notations and Definitions. For a set $E \subset \mathbb{R}^n$, $E^c = \mathbb{R}^n \setminus E$, \overline{E} stands for the closure of E , and $|E|$ is the Lebesgue measure of E .

2. TRUNCATED KERNELS AND OPERATORS

In this section, we study the truncated kernels and operators for $T \in CZO_\gamma$.

The following result can be proved with the same arguments as that in the proof of [11, Proposition 8.1.1].

Theorem 2.1. *Let $T \in CZO_\gamma$ and $\varepsilon > 0$. Set*

$$(2.1) \quad T_\varepsilon f = \int_{\rho(x,y) \geq \varepsilon} K(x,y) f(y) dy = \int_{\mathbb{R}^n} K_\varepsilon(x,y) f(y) dy,$$

where $K_\varepsilon(x,y) = K(x,y) \chi_{\rho(x,y) \geq \varepsilon}$. Assume that there exists a constant $B < \infty$ such that

$$\sup_{\varepsilon > 0} \|T_\varepsilon\|_{L^2 \rightarrow L^2} \leq B.$$

Then there exists a linear operator T_0 defined on $L^2(\mathbb{R}^n)$ such that

- (i) The Schwartz kernel of T_0 coincides with K on Γ^c .
- (ii) There exists some sequence $\varepsilon_j \downarrow 0$ such that

$$\int_{\mathbb{R}^n} (T_{\varepsilon_j} f)(x) g(x) dx \rightarrow \int_{\mathbb{R}^n} (T_0 f)(x) g(x) dx$$

as $j \rightarrow \infty$ for all $f, g \in L^2(\mathbb{R}^n)$.

- (iii) T_0 is bounded on $L^2(\mathbb{R}^n)$ with norm $\|T_0\|_{L^2 \rightarrow L^2} \leq B$.

To consider the difference between T and T_0 , we need more information on the set Γ .

Let $\mathcal{D} \subset \mathbb{R}^n$ and $\{I_j : j \in J\}$ be a sequence of subsets of \mathcal{D} . We say that $\{I_j : j \in J\}$ forms a partition of \mathcal{D} if $|I_j \cap I_{j'}| = 0$ for $j \neq j'$ and $|\mathcal{D} \setminus \bigcup_{j \in J} I_j| = 0$.

Lemma 2.2. *There is a sequence of closed cubes $\{I_j : j \in J\}$ which forms a partition of $\bigcup_{i=1}^r \gamma_i(\mathcal{D}_i)$ such that $\gamma_i^{-1}(I_j) \cap \gamma_{i'}^{-1}(I_j) = \emptyset$ for any $j \in J$ and $1 \leq i < i' \leq r$.*

Proof. Let $\mathcal{Y} = \bigcup_{i=1}^r \gamma_i(\mathcal{D}_i)$ and $Y = \{y \in \mathbb{R}^n : \text{there exist } 1 \leq i < i' \leq r \text{ and } x \in \mathbb{R}^n \text{ such that } y = \gamma_i(x) = \gamma_{i'}(x)\}$. Then Y has only finitely many elements.

Since every γ_i is continuous, for each y in the interior of $\mathcal{Y} \setminus Y$, there is some cube of the form $I_y := \prod_{k=1}^n [\frac{l_k}{2^m}, \frac{l_k+1}{2^m}]$, where $m, l_k \in \mathbb{Z}$ and $m > 0$, such that $y \in I_y \subset \mathcal{Y}$ and $\gamma_i^{-1}(I_y) \cap \gamma_{i'}^{-1}(I_y) = \emptyset$ for any $1 \leq i < i' \leq r$.

It follows that for different I_y 's, either they are mutually disjoint or one is contained in the other. Since $\mathcal{Y} \setminus Y = \bigcup_{y \in \mathcal{Y} \setminus Y} I_y$ and $\{I_y : y \in Y\}$ is at most countable, we get the conclusion. \square

Theorem 2.3. *Let the hypotheses be as in Theorem 2.1. Then there exist measurable functions b_i on \mathcal{D}_i such that*

$$(2.2) \quad (T - T_0)f(x) = \sum_{i=1}^r b_i(x) f(\gamma_i(x)) \chi_{\mathcal{D}_i}(x), \quad a.e.$$

for all $f \in L^2(\mathbb{R}^n)$. Moreover, $|b_i(x)|^2 \cdot |J_{\gamma_i}^{-1}(x)| \chi_{\mathcal{D}_i}(x) \in L^\infty$, $1 \leq i \leq r$.

Proof. We use notations in Theorem 2.1. By Lemma 2.2, there is a sequence of closed cubes $\{I_j : j \in J\}$ which forms a partition of $\bigcup_{i=1}^r \gamma_i(\mathcal{D}_i)$ and satisfies $\gamma_i^{-1}(I_j) \cap \gamma_{i'}^{-1}(I_j) = \emptyset$ for any $j \in J$ and $1 \leq i < i' \leq r$.

First, we prove that for $j \in J$,

$$(2.3) \quad (T_0 - T)(gf)(x) = (T_0 - T)(g)(x) \cdot f(\tilde{\gamma}_j(x)) \chi_{\tilde{\gamma}_j^{-1}(I_j)}(x),$$

where $f \in L^2(\mathbb{R}^n)$, g is bounded, $\text{supp } f, \text{supp } g \subset I_j$, and $\tilde{\gamma}_j$ is a mapping from $\gamma^{-1}(I_j)$ to I_j which is induced by γ , i.e., for $x \in \gamma_i^{-1}(I_j)$, $\tilde{\gamma}_j(x) = \gamma_i(x)$.

Fix some open cube $Q \subset I_j$. Observe that $\tilde{\gamma}_j^{-1}(Q) = \gamma^{-1}(Q)$ and $\overline{\gamma^{-1}(Q)} = \gamma^{-1}(\overline{Q})$. If $x \notin \overline{\tilde{\gamma}_j^{-1}(Q)}$, then $\varepsilon_Q := \min_{y \in \overline{Q}} \rho(x, y) > 0$. Consequently, for $0 < \varepsilon < \varepsilon_Q$, we have

$$(T_\varepsilon - T)(g\chi_Q)(x) = 0 = (T_\varepsilon - T)(g)(x) \cdot \chi_{\tilde{\gamma}_j^{-1}(Q)}(x).$$

On the other hand, if $x \in \tilde{\gamma}_j^{-1}(Q)$, then $x \notin \tilde{\gamma}_j^{-1}(I_j \setminus Q) = \gamma^{-1}(I_j \setminus Q)$. Therefore, $\min_{y \in I_j \setminus Q} \rho(x, y) > 0$. It follows that for ε sufficiently small, we have

$$(T_\varepsilon - T)(g\chi_{I_j \setminus Q})(x) = 0 = (T_\varepsilon - T)(g)(x) \cdot \chi_{\tilde{\gamma}_j^{-1}(I_j \setminus Q)}(x).$$

Consequently, for $x \in \tilde{\gamma}_j^{-1}(Q)$, we have

$$\begin{aligned} (T_\varepsilon - T)(g\chi_Q)(x) &= (T_\varepsilon - T)(g)(x) - (T_\varepsilon - T)(g\chi_{I_j \setminus Q})(x) \\ &= (T_\varepsilon - T)(g)(x) \cdot \chi_{\mathbb{R}^n \setminus \tilde{\gamma}_j^{-1}(I_j \setminus Q)}(x) \\ &= (T_\varepsilon - T)(g)(x) \cdot \chi_{\tilde{\gamma}_j^{-1}(Q)}(x). \end{aligned}$$

Summing up the above arguments, we get that for almost every x , whenever ε is sufficiently small,

$$\begin{aligned} (T_\varepsilon - T)(g\chi_Q)(x) &= (T_\varepsilon - T)(g)(x) \cdot \chi_{\tilde{\gamma}_j^{-1}(Q)}(x) \\ &= (T_\varepsilon - T)(g)(x) \cdot \chi_Q(\tilde{\gamma}_j(x))\chi_{\tilde{\gamma}_j^{-1}(I_j)}(x). \end{aligned}$$

Taking weak limits in the above equations, we get

$$(2.4) \quad (T_0 - T)(g\chi_Q)(x) = (T_0 - T)(g)(x) \cdot \chi_Q(\tilde{\gamma}_j(x))\chi_{\tilde{\gamma}_j^{-1}(I_j)}(x), \quad a.e.$$

By linearity, we extend (2.4) to simple functions, and then to arbitrary $f \in L^2(\mathbb{R}^n)$ which is supported in I_j , i.e.,

$$(2.5) \quad (T_0 - T)(gf)(x) = (T_0 - T)(g)(x) \cdot f(\tilde{\gamma}_j(x))\chi_{\tilde{\gamma}_j^{-1}(I_j)}(x), \quad a.e.$$

Assume that $I_j = \prod_{k=1}^n [a_{j,k}, b_{j,k}]$. For $t_k, t'_k \in [a_{j,k}, b_{j,k}]$ with $t_k < t'_k$, define $I_{j,t} = \prod_{k=1}^n [a_{j,k}, t_k]$ and $I_{j,t'} = \prod_{k=1}^n [a_{j,k}, t'_k]$. We have

$$\begin{aligned} (T_0 - T)(\chi_{I_{j,t}})(x) &= (T_0 - T)(\chi_{I_{j,t}} \cdot \chi_{I_{j,t'}})(x) \\ &= (T_0 - T)(\chi_{I_{j,t'}})(x)\chi_{I_{j,t}}(\tilde{\gamma}_j(x))\chi_{\tilde{\gamma}_j^{-1}(I_j)}(x). \end{aligned}$$

It follows that for $x \in \tilde{\gamma}_j^{-1}(I_{j,t})$, we have

$$(T_0 - T)(\chi_{I_{j,t}})(x) = (T_0 - T)(\chi_{I_{j,t'}})(x).$$

Consequently, there is a function h_j defined on $\tilde{\gamma}_j^{-1}(I_j)$ such that

$$h_j(x) = (T_0 - T)(\chi_{I_{j,t}})(x), \quad x \in \tilde{\gamma}_j^{-1}(I_{j,t}).$$

For any $f \in L^2(\mathbb{R}^n)$, let $f_j = f \cdot \chi_{I_j}$. Then we have

$$\begin{aligned} (T_0 - T)(f_j \cdot \chi_{I_{j,t}})(x) &= f_j(\tilde{\gamma}_j(x))(T_0 - T)(\chi_{I_{j,t}})(x)\chi_{\tilde{\gamma}_j^{-1}(I_j)}(x) \\ &= f_j(\tilde{\gamma}_j(x))h_j(x), \quad x \in \tilde{\gamma}_j^{-1}(I_{j,t}). \end{aligned}$$

By letting $(t_1, \dots, t_n) \rightarrow (b_{j,1}, \dots, b_{j,n})$, we get

$$(T_0 - T)(f_j)(x) = f_j(\tilde{\gamma}_j(x))h_j(x), \quad x \in \tilde{\gamma}_j^{-1}(I_j).$$

Observe that $(T_0 - T)(f_j)(x) = 0$ for $x \notin \tilde{\gamma}_j^{-1}(I_j)$. We have

$$(2.6) \quad (T_0 - T)(f_j)(x) = f_j(\tilde{\gamma}_j(x))h_j(x)\chi_{\tilde{\gamma}_j^{-1}(I_j)}(x), \quad a.e.$$

Since $(T_0 - T)(f)(x) = 0$ whenever $\text{supp } f \subset (\bigcup_{i=1}^r \gamma_i(\mathcal{D}_i))^c$, we have

$$(T_0 - T)(f)(x) = \sum_{j \in J} f_j(\tilde{\gamma}_j(x))h_j(x)\chi_{\tilde{\gamma}_j^{-1}(I_j)}(x), \quad a.e.$$

Note that for every $j \in J$, we have $\tilde{\gamma}_j^{-1}(I_j) = \bigcup_{i=1}^r \gamma_i^{-1}(I_j)$. Hence

$$\begin{aligned} (T_0 - T)(f)(x) &= \sum_{j \in J} f_j(\tilde{\gamma}_j(x))h_j(x) \sum_{i=1}^r \chi_{\gamma_i^{-1}(I_j)}(x) \\ &= \sum_{i=1}^r \sum_{j \in J} f_j(\tilde{\gamma}_j(x))h_j(x)\chi_{\gamma_i^{-1}(I_j)}(x) \\ &= \sum_{i=1}^r f(\gamma_i(x)) \sum_{j \in J} h_j(x)\chi_{\gamma_i^{-1}(I_j)}(x). \end{aligned}$$

By setting $b_i(x) = \sum_{j \in J} h_j(x)\chi_{\gamma_i^{-1}(I_j)}(x)$, we get (2.2).

Take some $f \in L^2(\mathbb{R}^n)$ and $j \in J$. By (2.6), we have

$$\begin{aligned} \|(T_0 - T)(f \cdot \chi_{I_j})\|_2^2 &= \left\| \sum_{i=1}^r f(\gamma_i(x))b_i(x)\chi_{\gamma_i^{-1}(I_j)}(x) \right\|_2^2 \\ &= \sum_{i=1}^r \int_{\gamma_i^{-1}(I_j)} |f(\gamma_i(x))b_i(x)|^2 dx \\ &= \sum_{i=1}^r \int_{I_j} |f(y)b_i(\gamma_i^{-1}(y))|^2 |J_{\gamma_i^{-1}}(y)| dy. \end{aligned}$$

Since $\|(T_0 - T)(f \cdot \chi_{I_j})\|_2^2 \leq (\|T\|_{L^2 \rightarrow L^2} + B)^2 \|f \cdot \chi_{I_j}\|_2^2$ for any $f \in L^2(\mathbb{R}^n)$, we have

$$\sum_{i=1}^r |b_i(\gamma_i^{-1}(y))|^2 |J_{\gamma_i^{-1}}(y)| \leq (\|T\|_{L^2 \rightarrow L^2} + B)^2, \quad a.e. \text{ on } I_j.$$

Hence

$$\sum_{i=1}^r |b_i(x)|^2 |J_{\gamma_i^{-1}}(x)| \leq (\|T\|_{L^2 \rightarrow L^2} + B)^2, \quad a.e. \text{ on } \gamma_i^{-1}(I_j).$$

Therefore, $|b_i(x)|^2 |J_{\gamma_i^{-1}}(x)| \cdot \chi_{\mathcal{D}_i}(x) \in L^\infty$, $1 \leq i \leq r$. This completes the proof. \square

The following is an immediate consequence, which gives the difference between two operators which share the same kernel.

Corollary 2.4. *Let S and T be two operators in CZO_γ which share the same kernel K . Then there exist measurable functions b_i on \mathcal{D}_i such that*

$$(2.7) \quad (S - T)f(x) = \sum_{i=1}^r b_i(x)f(\gamma_i(x))\chi_{\mathcal{D}_i}(x), \quad a.e.$$

for all $f \in L^2(\mathbb{R}^n)$. Moreover, $|b_i(x)|^2 \cdot |J_{\gamma_i^{-1}}(x)|\chi_{\mathcal{D}_i}(x) \in L^\infty$, $1 \leq i \leq r$.

Proof. Let $(S - T)_\varepsilon$ be defined similarly as T_ε in (2.1). Since $S - T$ has the kernel zero, we have $(S - T)_0 := \lim_{\varepsilon \rightarrow 0} (S - T)_\varepsilon = 0$. Consequently,

$$S - T = S - T - (S - T)_0.$$

Now the conclusion follows by Theorem 2.3. \square

Now we give two examples. In the first example, Γ consists of two hyper curves which have one common point.

Example 2.5. let $\gamma(x) = \pm x$. It is easy to check that γ determines a standard hyper curve in $\mathbb{R}^n \times \mathbb{R}^n$. By Theorem 2.3, for any $T \in CZO_\gamma$, we can find $b_1, b_2 \in L^\infty$ such that

$$(T - T_0)(f)(x) = b_1(x)f(x) + b_2(x)f(-x), \quad x \in \mathbb{R}^n,$$

And in the next example, Γ is a diamond.

Example 2.6. Suppose that $n = 1$. Let $\gamma(x) = \pm(1 - |x|)$ for $|x| \leq 1$, and 0 for $|x| \geq 1$. Then γ determines a standard hyper curve in $\mathbb{R} \times \mathbb{R}$. By Theorem 2.3, for any $T \in CZO_\gamma$, we can find $b_1, b_2 \in L^\infty$ such that

$$(T - T_0)(f)(x) = \begin{cases} b_1(x)f(1 - |x|) + b_2(x)f(|x| - 1), & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

3. WEAK TYPE (1,1) AND THE L^p BOUNDEDNESS

It is well known that a Calderón-Zygmund operator is of weak type $(1, 1)$ and strong type (p, p) for $1 < p < \infty$. In this section, we show that operators in CZO_γ have the same property.

First, we give some properties of $\rho(x, y)$.

For any $x, y \in \mathbb{R}^n$, let $\xi_{i,x}$ be the point in \mathcal{D}_i which is most close to x , i.e.,

$$\xi_{i,x} = \arg \min_{x' \in \mathcal{D}_i} |x - x'|.$$

Similarly,

$$\eta_{i,y} = \arg \min_{y' \in \gamma_i(\mathcal{D}_i)} |y - y'|.$$

For a cube Q , we define $\eta_{i,Q} = \{\eta_{i,y} : y \in Q\}$.

For any $x, y \in \mathbb{R}^n$, let

$$(3.1) \quad \tilde{\rho}_i(x, y) = |x - \xi_{i,x}| + |y - \gamma_i(\xi_{i,x})|,$$

$$(3.2) \quad \tilde{\rho}_i^*(x, y) = |y - \eta_{i,y}| + |x - \gamma_i^{-1}(\eta_{i,y})|,$$

$$\tilde{\rho}(x, y) = \min_{1 \leq i \leq r} \tilde{\rho}_i(x, y) \text{ and } \tilde{\rho}^*(x, y) = \min_{1 \leq i \leq r} \tilde{\rho}_i^*(x, y).$$

Next we show that both $\tilde{\rho}(x, y)$ and $\tilde{\rho}^*(x, y)$ are equivalent to $\rho(x, y)$.

Lemma 3.1. For any $x, y \in \mathbb{R}^n$, we have

$$(3.3) \quad \rho_i(x, y) \leq \tilde{\rho}_i(x, y) \leq 2(c_\gamma + 1)\rho_i(x, y),$$

$$(3.4) \quad \rho_i(x, y) \leq \tilde{\rho}_i^*(x, y) \leq 2(c_\gamma + 1)\rho_i(x, y).$$

Proof. Fix some (x, y) . There exists some $x_0 \in \mathcal{D}_i$ such that $\rho_i(x, y) = |(x, y) - (x_0, \gamma_i(x_0))|$.

If $x \in \mathcal{D}_i$, then $\xi_{i,x} = x$. It follows that

$$\begin{aligned} \rho_i(x, y) &\leq |y - \gamma_i(x)| \\ &\leq |y - \gamma_i(x_0)| + |\gamma_i(x_0) - \gamma_i(x)| \\ &\leq |y - \gamma_i(x_0)| + c_\gamma |x_0 - x| \end{aligned}$$

$$\leq 2^{1/2} c_\gamma \rho_i(x, y).$$

If $x \notin \mathcal{D}_i$, then we have

$$\begin{aligned} \rho_i(x, y) &\leq |x - \xi_{i,x}| + |y - \gamma_i(\xi_{i,x})| \\ &\leq |x - x_0| + |y - \gamma_i(x_0)| + |\gamma_i(x_0) - \gamma_i(\xi_{i,x})| \\ &\leq 2^{1/2} \rho_i(x, y) + c_\gamma |x_0 - \xi_{i,x}| \\ &\leq 2^{1/2} \rho_i(x, y) + c_\gamma (|x_0 - x| + |x - \xi_{i,x}|) \\ &\leq 2^{1/2} \rho_i(x, y) + 2c_\gamma |x_0 - x| \\ &\leq (2c_\gamma + 2) \rho_i(x, y). \end{aligned}$$

This proves (3.3). And (3.4) can be proved similarly. \square

We see from Lemma 3.1 that

$$(3.5) \quad \rho(x, y) \leq \tilde{\rho}(x, y) \leq 2(c_\gamma + 1) \rho(x, y),$$

$$(3.6) \quad \rho(x, y) \leq \tilde{\rho}^*(x, y) \leq 2(c_\gamma + 1) \rho(x, y).$$

In other words, all of $\rho(x, y)$, $\tilde{\rho}(x, y)$ and $\tilde{\rho}^*(x, y)$ are equivalent.

With the result above, we can prove that (1.5) and (1.6) implies the Hörmander condition.

Lemma 3.2 (The Hörmander condition). *Suppose that the kernel $K(x, y)$ satisfies (1.5) and (1.6). Then there is some constant C such that*

$$(3.7) \quad \int_{\rho(x,y) \geq 2|y-z|} |K(x, y) - K(x, z)| dx \leq C,$$

$$(3.8) \quad \int_{\rho(y,x) \geq 2|y-z|} |K(y, x) - K(z, x)| dx \leq C.$$

Proof. We only need to prove (3.7). And (3.8) can be proved similarly.

Set $a = |y - z|$. By Lemma 3.1, we have

$$\begin{aligned} &\int_{\rho(x,y) \geq 2|y-z|} |K(x, y) - K(x, z)| dx \\ &\leq \int_{\rho(x,y) \geq 2a} \frac{Aa^\delta}{\rho(x, y)^{n+\delta}} dx \\ &\leq \int_{\tilde{\rho}^*(x,y) \geq 2a} \frac{A(2c_\gamma + 2)^{n+\delta} a^\delta}{\tilde{\rho}^*(x, y)^{n+\delta}} dx \\ &\leq \sum_{i=1}^r \int_{\tilde{\rho}_i^*(x,y) = \tilde{\rho}^*(x,y) \geq 2a} \frac{C' a^\delta}{\tilde{\rho}_i^*(x, y)^{n+\delta}} dx \\ &\leq \sum_{i=1}^r \left(\int_{\tilde{\rho}_i^*(x,y) \geq 2a, |x - \gamma_i^{-1}(\eta_{i,y})| \leq a} \frac{C' a^\delta}{\tilde{\rho}_i^*(x, y)^{n+\delta}} dx + \int_{\tilde{\rho}_i^*(x,y) \geq 2a, |x - \gamma_i^{-1}(\eta_{i,y})| \geq a} \frac{C' a^\delta}{\tilde{\rho}_i^*(x, y)^{n+\delta}} dx \right) \\ &\leq \sum_{i=1}^r \left(C'' + \int_{|x - \gamma_i^{-1}(\eta_{i,y})| \geq a} \frac{C' a^\delta}{|x - \gamma_i^{-1}(\eta_{i,y})|^{n+\delta}} dx \right) \\ &\leq C. \end{aligned}$$

This completes the proof. \square

Given a positive number θ and a cube $Q \subset \mathbb{R}^n$, we define

$$Q_{i,\theta} = \begin{cases} \{x : d(x, \gamma_i^{-1}(\eta_{i,Q})) \leq \theta \cdot \ell(Q)\}, & d(Q, \gamma_i(\mathcal{D}_i)) < 2n^{1/2}\ell(Q), \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\ell(Q)$ is the side length of Q . Let $Q_\theta = \bigcup_{1 \leq i \leq r} Q_{i,\theta}$.

The following result is needed in the proof of weak type $(1, 1)$.

Lemma 3.3. *Let $\Gamma \in SHC$.*

- (i) *There is some constant C such that for any $\theta > 1$ and cube $Q \subset \mathbb{R}^n$, $|Q_\theta| \leq C\theta^n|Q|$.*
- (ii) *For $\theta > 2n^{1/2} + 5n^{1/2}c_\gamma$ and every cube Q , we have $\rho(x, y) \geq 2n^{1/2}\ell(Q)$ for all $x \notin Q_\theta$ and $y \in Q$.*

Proof. First, we prove (i). For each $1 \leq i \leq r$, we only need to consider the case $d(Q, \gamma_i(\mathcal{D}_i)) < 2n^{1/2}\ell(Q)$. In this case, there is some $y_i \in Q$ such that

$$d(y_i, \gamma_i(\mathcal{D}_i)) = d(Q, \gamma_i(\mathcal{D}_i)) < 2n^{1/2}\ell(Q).$$

For any $x \in Q_{i,\theta}$, there is some $y_x \in Q$ such that

$$|x - \gamma_i^{-1}(\eta_{i,y_x})| \leq \theta \cdot \ell(Q).$$

It follows that

$$\begin{aligned} |x - \gamma_i^{-1}(\eta_{i,y_i})| &\leq |x - \gamma_i^{-1}(\eta_{i,y_x})| + |\gamma_i^{-1}(\eta_{i,y_x}) - \gamma_i^{-1}(\eta_{i,y_i})| \\ &\leq \theta \cdot \ell(Q) + c_\gamma |\eta_{i,y_i} - \eta_{i,y_x}| \\ &\leq \theta \cdot \ell(Q) + c_\gamma (|\eta_{i,y_i} - y_i| + |y_i - y_x| + |y_x - \eta_{i,y_x}|) \\ &= \theta \cdot \ell(Q) + c_\gamma (d(y_i, \gamma_i(\mathcal{D}_i)) + |y_i - y_x| + d(y_x, \gamma_i(\mathcal{D}_i))) \\ &\leq \theta \cdot \ell(Q) + 2c_\gamma (d(y_i, \gamma_i(\mathcal{D}_i)) + |y_i - y_x|) \\ &\leq (\theta + 6n^{1/2}c_\gamma) \ell(Q). \end{aligned}$$

Hence

$$Q_\theta \subset \bigcup_i \left\{ x : |x - \gamma_i^{-1}(\eta_{i,y_i})| \leq (\theta + 6n^{1/2}c_\gamma) \ell(Q) \right\}.$$

Since $\theta > 1$, we have $|Q_\theta| \leq C\theta^n|Q|$.

Next, we prove (ii). Fix some $1 \leq i \leq r$. There are two cases.

Case 1. $d(Q, \gamma_i(\mathcal{D}_i)) \geq 2n^{1/2}\ell(Q)$. In this case, for any $x \in \mathbb{R}^n$ and $y \in Q$,

$$\rho_i(x, y) = \min_{z \in \gamma_i(\mathcal{D}_i)} |(x, y) - (\gamma_i^{-1}(z), z)| \geq \min_{z \in \gamma_i(\mathcal{D}_i)} |y - z| \geq 2n^{1/2}\ell(Q).$$

Case 2. $d(Q, \gamma_i(\mathcal{D}_i)) < 2n^{1/2}\ell(Q)$. We conclude that for any $x \notin Q_\theta$ and $z \in \gamma_i(\mathcal{D}_i)$, $\max\{|x - \gamma_i^{-1}(z)|, |y - z|\} \geq 2n^{1/2}\ell(Q)$.

To see this, take some $y_i \in Q$ such that $d(y_i, \gamma_i(\mathcal{D}_i)) = d(Q, \gamma_i(\mathcal{D}_i)) < 2n^{1/2}\ell(Q)$. If $|y - z| \leq 2n^{1/2}\ell(Q)$, then we have

$$\begin{aligned} |x - \gamma_i^{-1}(z)| &\geq |x - \gamma_i^{-1}(\eta_{i,y})| - |\gamma_i^{-1}(\eta_{i,y}) - \gamma_i^{-1}(z)| \\ &\geq \theta \cdot \ell(Q) - c_\gamma |\eta_{i,y} - z| \\ &\geq \theta \cdot \ell(Q) - c_\gamma (|\eta_{i,y} - y| + |y - z|) \\ &\geq \theta \cdot \ell(Q) - c_\gamma (|y - y_i| + d(y_i, \gamma_i(\mathcal{D}_i)) + |y - z|) \end{aligned}$$

$$\begin{aligned} &\geq (\theta - 5n^{1/2}c_\gamma)\ell(Q) \\ &\geq 2n^{1/2}\ell(Q). \end{aligned}$$

Hence $\max\{|x - \gamma_i^{-1}(z)|, |y - z|\} \geq 2n^{1/2}\ell(Q)$. Therefore,

$$\rho_i(x, y) = \min_{z \in \gamma_i(\mathcal{D}_i)} |(x, y) - (\gamma_i^{-1}(z), z)| \geq 2n^{1/2}\ell(Q).$$

In both cases, we get $\rho_i(x, y) \geq 2n^{1/2}\ell(Q)$. It follows that $\rho(x, y) \geq 2n^{1/2}\ell(Q)$ for all $x \notin Q_\theta$ and $y \in Q$. This completes the proof. \square

The weak type $(1, 1)$ and the L^p boundedness for $T \in CZO_\gamma$ can be proved similarly to the one for ordinary Calderón-Zygmund operators. To keep the paper more readable, we include a proof here.

Theorem 3.4. *Let $T \in CZO_\gamma$ and $1 < p < \infty$. Then there is some constant $C > 0$ such that*

$$(3.9) \quad \|Tf\|_{L^{1,\infty}} \leq C\|f\|_{L^1}$$

and

$$(3.10) \quad \|Tf\|_{L^p} \leq C\|f\|_{L^p}.$$

Proof. First, we show that T is weakly bounded on a dense set of $L^1(\mathbb{R}^n)$. Fix some $f \in L^1 \cap L^2(\mathbb{R}^n)$ and $\lambda > 0$. Form the Calderón-Zygmund decomposition of f at height λ . We get disjoint cubes $\{Q_k : k \in \mathbb{K}\}$ such that

$$|f(x)| \leq \lambda, \quad x \notin \bigcup_{k \in \mathbb{K}} Q_k$$

and

$$\lambda \leq \frac{1}{|Q_k|} \int_{Q_k} |f(x)| dx \leq 2^n \lambda, \quad k \in \mathbb{K}.$$

Define

$$g(x) = \begin{cases} f(x), & x \notin \bigcup_{k \in \mathbb{K}} Q_k, \\ |Q_k|^{-1} \int_{Q_k} f(y) dy, & x \in \text{interior of } Q_k, \end{cases}$$

and

$$b_k(x) = \begin{cases} 0, & x \notin Q_k, \\ f(x) - |Q_k|^{-1} \int_{Q_k} f(y) dy, & x \in \text{interior of } Q_k. \end{cases}$$

Since Q_k are disjoint, the series $b(x) := \sum_{k \in \mathbb{K}} b_k(x)$ is convergent in $L^2(\mathbb{R}^n)$. Moreover, for any $\lambda > 0$,

$$(3.11) \quad \{x : |(Tf)(x)| \geq \lambda\} \subset \{x : |(Tg)(x)| \geq \frac{\lambda}{2}\} \bigcup \{x : |(Tb)(x)| \geq \frac{\lambda}{2}\}.$$

Take some $\theta > 2n^{1/2} + 5n^{1/2}c_\gamma$. We see from Lemma 3.3 that for any cube Q , $\rho(x, y) \geq 2|y - z|$ for all $x \notin Q_\theta$ and $y, z \in Q$.

For each $k \in \mathbb{K}$, set $Q_k^* = (Q_k)_\theta$. Let $B^* = \bigcup_{k \in \mathbb{K}} Q_k^*$ and $G^* = \mathbb{R}^n \setminus B^*$. Then we have

$$\begin{aligned} (3.12) \quad \left| \{x : |Tg(x)| \geq \frac{\lambda}{2}\} \right| &\leq \frac{4}{\lambda^2} \int_{\{x : |Tg(x)| \geq \frac{\lambda}{2}\}} |Tg(x)|^2 dx \\ &\leq \frac{4\|T\|_{L^2 \rightarrow L^2}^2}{\lambda^2} \int_{\mathbb{R}^n} |g(x)|^2 dx \\ &\leq \frac{4\|T\|_{L^2 \rightarrow L^2}^2}{\lambda^2} \left(\int_{G^*} |g(x)|^2 dx + \sum_{k \in \mathbb{K}} \int_{Q_k} 2^n \lambda |g(x)| dx \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{4\|T\|_{L^2 \rightarrow L^2}^2}{\lambda^2} \left(\int_G \lambda |f(x)| dx + 2^n \lambda \sum_{k \in \mathbb{K}} \int_{Q_k} |f(x)| dx \right) \\
&\leq \frac{2^{n+2}}{\lambda} \|T\|_{L^2 \rightarrow L^2}^2 \|f\|_{L^1}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(3.13) \quad \left| \left\{ x : |Tb(x)| \geq \frac{\lambda}{2} \right\} \right| &\leq |B^*| + \left| \left\{ x \in G^* : |Tb(x)| \geq \frac{\lambda}{2} \right\} \right| \\
&\leq C' \theta^n \left| \bigcup_{k \in \mathbb{K}} Q_k \right| + \frac{2}{\lambda} \int_{G^*} |Tb(x)| dx \\
&\leq \frac{C' \theta^n}{\lambda} \|f\|_1 + \frac{2}{\lambda} \sum_{k \in \mathbb{K}} \int_{G^*} |Tb_k(x)| dx.
\end{aligned}$$

Observe that

$$\begin{aligned}
(3.14) \quad \int_{G^*} |Tb_k(x)| dx &\leq \int_{\mathbb{R}^n \setminus Q_k^*} |Tb_k(x)| dx \\
&= \int_{\mathbb{R}^n \setminus Q_k^*} dx \left| \int_{Q_k} K(x, y) b_k(y) dy \right| \\
&= \int_{\mathbb{R}^n \setminus Q_k^*} dx \left| \int_{Q_k} (K(x, y) - K(x, y_k)) b_k(y) dy \right| \\
&\leq \int_{Q_k} |b_k(y)| dy \int_{\rho(x, y) \geq 2|y - y_k|} |K(x, y) - K(x, y_k)| dx \\
&\leq C \int_{Q_k} |b_k(y)| dy,
\end{aligned}$$

where y_k is the center of Q_k and we use Lemma 3.2 in the last step. Now we see from (3.14) that

$$\begin{aligned}
(3.15) \quad \sum_{k \in \mathbb{K}} \int_{G^*} |Tb_k(x)| dx &\leq C \sum_{k \in \mathbb{K}} \int_{Q_k} |b_k(y)| dy \\
&\leq C \sum_{k \in \mathbb{K}} \int_{Q_k} \left(|f(y)| + \frac{1}{|Q_k|} \int_{Q_k} |f(x)| dx \right) dy \\
&= 2C \sum_{k \in \mathbb{K}} \int_{Q_k} |f(y)| dy \\
&\leq 2C \|f\|_{L^1}.
\end{aligned}$$

Since $f(x) = g(x) + b(x)$, together with (3.12), (3.13) and (3.15), we have

$$(3.16) \quad \|Tf\|_{L^1_{weak}} \leq (2^{n+2} \|T\|_{L^2 \rightarrow L^2}^2 + C' \theta^n + 4C) \|f\|_1,$$

where $f \in L^1 \cap L^2(\mathbb{R}^n)$.

For any $f \in L^1(\mathbb{R}^n)$, we can find a sequence $\{f_k : k \geq 1\} \subset L^1 \cap L^2(\mathbb{R}^n)$ such that $\|f - f_k\|_1 \rightarrow 0$. By (3.16), we have

$$\left| \left\{ x : |(T(f_k - f_{k'}))(x)| \geq \lambda \right\} \right| \leq \frac{C''}{\lambda} \|f_k - f_{k'}\|_1.$$

Hence $\{Tf_k : k \geq 1\}$ is convergent in measure. Since $\{f_k : k \geq 1\}$ is arbitrary, it is easy to see that the limit, denoted by Tf , is independent of choices of $\{f_k : k \geq 1\}$. Hence, (3.16) is true for any $f \in L^1(\mathbb{R}^n)$. This proves (3.9).

By the interpolation theorem, we prove (3.10) with $1 < p < 2$. And by a duality argument, we prove (3.10) with $p > 2$. \square

We see from Theorem 3.4 that an operator $T \in CZO_\gamma$ is well defined on $L^p(\mathbb{R})$ for $1 \leq p < \infty$. As for ordinary Calderón-Zygmund operators, Tf can also be expressed as integrals in some cases. Specifically, we have the following result, which can be proved similarly to [11, Proposition 8.2.2].

Theorem 3.5. *Let T be an operator in CZO_γ associated with some kernel K . Then for $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the following absolutely convergent integral representation is valid:*

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy, \quad \text{a.e. on } \gamma^{-1}(\text{supp } f).$$

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